"FAILURE IS NOT AN OPTION BUT IT IS A PREVAILING ATTITUDE"

ALGEBRA

1. The remainder when the expression $x^3 - 2x^2 + ax + b$ is divided by $x - 2$ is five times the remainder when the same expression is divided by $x - 1$, and 12 less than the remainder when the same expression is divided by $x - 3$. Find the values of a and b.

2. Solve the equation: $\log_{10} e.In(x^2 + 1) - 2\log_{10} e.Inx = \log_{10} 5$

$$\frac{Ine}{In10} In(x^2 + 1) - 2\frac{Ine}{In10} In(x) = \frac{In5}{In10}$$

$$In\frac{x^2 + 1}{x^2} = In5 \qquad \text{Thus, } \frac{x^2 + 1}{x^2} = 5$$

$$4x^2 = 1 \qquad x^2 = \frac{1}{4}, \Rightarrow \quad x = \pm\frac{1}{2} \qquad \therefore x = \frac{1}{2}$$

3. Solve for x : $\frac{(1 - x)^2}{2 - x^2} = \frac{(1 - a)^2}{2 - a^2}$

$$\frac{1 - 2x + x^2}{2 - x^2} = \frac{1 - 2a + a^2}{2 - a^2}, \quad (1 - 2x + x^2)(2 - a^2) = (1 - 2a + a^2)(2 - x^2)$$

$$2 - a^2 - 4x + 2a^2x + 2x^2 - a^2x^2 = 2 - 4a + 2a^2 - x^2 + 2ax^2 - a^2x^2$$

$4(x - a) - 3(x^2 - a^2) + 2ax(x - a) = 0$ (factorise by grouping)

$$(x - a)[4 - 3(x + a) + 2ax] = 0$$

Either $x = a$ Or $4 - 3x - 3a + 2ax = 0$ So, $x = \frac{3a - 4}{2a - 3}$

4. Find x if $\log_5 2,\ \log_5(2^x - 3),\ \log_5(\frac{17}{2} + 2^{x-1})$ form an A.P.

5. The first, second, third and n th terms of a series are 4, -3, -16 and $(an^2 + bn + c)$ respectively. Find a, b, c and the sum of n terms of the series.

$$n = 1, \quad a + b + c = 0$$

$$n = 2, \quad 4a + 2b + c = -3$$

$$n = 3, \quad 9a + 3b + c = -16$$

Eliminate c, $5a + b = -13$, $\qquad 3a + b = -7$

Eliminate b, $2a = -6$, $\therefore\ a = -3 \quad b = 2 \quad c = 5$

$\therefore\ a_n = 3n^2 + 2n + 5$

$(-3\ x1^2 + 2\ x1 + 5) + (-3\ x\ 2^2\ x\ 2\ x\ 2 + 5) + \ldots + (-3n^2 + 2n + 5)$

$$\begin{aligned} & (-3\ x1^2 + 2\ x1 + 5) \\ + & (-3\ x\ 2^2 + 2\ x\ 2 + 5) \\ + & \ldots \qquad\qquad \ldots \\ + & (-3\ x\ n^2 + 2\ x\ n + 5) \end{aligned}$$

$$= -3\left(\frac{1}{6}n(n+1)(2n+1)\right) + 2\left(\frac{1}{2}n(n+1)\right) + 5n$$

$$= \frac{1}{2}(-2n^3 - 3n^2 - n + 2n^2 + 2n + 10n) \qquad = -\frac{n}{2}(2n^2 + n - 11)$$

6. The coefficients of the 5th, 6th and 7th terms in the expansion of $(1 + x)^n$ are in an A.P. Find n.

Coefficient of $T_5 = {}^nC_4$, Coefficient of $T_6 = {}^nC_5$, and Coefficient of $T_7 = {}^nC_6$

Since these are in an A.P, then, we have ${}^nC_5 - {}^nC_4 = {}^nC_6 - {}^nC_5$

$$\Rightarrow \frac{n!}{5!(n-5)!} - \frac{n!}{4!(n-4)!} = \frac{n!}{6!(n-6)!} - \frac{n!}{5!(n-5)!}$$

$$\Rightarrow \frac{1}{120(n-5)!} - \frac{1}{24(n-4)\times(n-5)!} = \frac{1}{720(n-6)!} - \frac{1}{120(n-5)(n-6)!}$$

$$\Rightarrow \frac{1}{24(n-5)!}\left[\frac{1}{5} - \frac{1}{n-4}\right] = \frac{1}{120(n-6)!}\left[\frac{1}{6} - \frac{1}{n-5}\right]$$

$$\Rightarrow \frac{1}{(n-5)}\left[\frac{n-4-5}{5(n-4)}\right] = \frac{1}{5}\left[\frac{n-5-6}{6(n-5)}\right]$$

$$\frac{n-9}{n-4} = \frac{n-11}{6}, \quad \Rightarrow n^2 - 21n + 98 = 0$$

$$(n-7)(n-14) = 0 \quad \therefore n = 7, \quad n = 14$$

7. Given that the first three terms in the expansion in ascending powers of x of

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$(1+x+x^2)^n$ are the same as the first three terms in the expansion of $\left(\frac{1+ax}{1-3ax}\right)^3$, find the value of a and n.

$$(1+x+x^2)^n = 1 + n(x+x^2) + \frac{n(n-1)(x+x^2)^2}{2!} + \ldots$$

$$= 1 + nx + nx^2 + \frac{1}{2}n(n-1)x^2 + \ldots$$

$$\left(\frac{1+ax}{1-3ax}\right)^3 = (1+ax)^3(1-3ax)^{-3}$$

$$= (1+3ax+3a^2x^2+\ldots)(1+9ax+18a^2x^2+\ldots)$$

$$= 1+9ax+54a^2x^2+3ax+27a^2x^2+3a^2x^2+\ldots$$

$$= 1+12ax+84a^2x^2+\ldots$$

Thus $1+nx+nx^2+\frac{1}{2}n(n-1)x^2+\ldots = 1+12ax+84a^2x^2+\ldots$

Equating coefficients, we get $n = 12a \ldots\ldots\ldots\ \ldots(i)$

$$84a^2 = n + \frac{n(n-1)}{2} \ldots\ldots\ldots\ \ldots(ii)$$

$$84a^2 = 12a + \frac{12a(12a-1)}{2}, \quad 84a^2 = 12a + 72a^2 - 6a$$

$12a^2 - 6a = 0$, for $a \neq 0$, $a = \frac{1}{2}$ thus $n = 6$

8. Prove by induction that $8^n - 7n + 6$ is divisible by 7 for all $n \geq 1$.

Let $a_n = 8^n - 7n + 6$

For $n = 1$, $a_1 = 8 - 7 + 6 = 7$, divisible by 7.

For $n = 2$, $a_1 = 64 - 14 + 6 = 56$, divisible by 7.

For $n = k$, $a_k = 8^k - 7k + 6$,

When $n = k+1, \quad a_{k+1} = 8^{k+1} - 7(k+1) + 6$

Thus $a_{k+1} - a_k = 8^{k+1} - 7(k+1) + 6 - (8^k - 7k + 6)$

$= 8^k . 8 - 8^k - 7k + 7k - 7 + 6 - 6$

$= 7.8^k - 7 = 7(8^k - 1)$ which is still divisible by 8. Thus if its true for $n = 1, 2, ..., k, k+1$, then $8^n - 7n + 6$ is divisible by 7 for all $n \geq 1$.

9. Given that $z + 2i$ is a factor of $z^4 + 2z^3 + 7z^2 + 8z + 12$, solve the equation $z^4 + 2z^3 + 7z^2 + 8z + 12 = 0$.

If $z + 2i$ is a factor, then, $z - 2i$, thus the quadratic factor is $z^2 + 4$

$$\begin{array}{r} z^2 + 2z + 3 \\ z^2 + 4 \overline{\big) z^4 + 2z^3 + 7z^2 + 8z + 12} \\ z^4 + \quad 4z^2 \end{array}$$

By long division,

Thus, $(z^4 + 4)(z^2 + 2z + 3) = 0$

$z = \pm 2i, \quad z = \dfrac{-2 \pm \sqrt{-8}}{2}$

The roots are $2i, \ -2i, \ -1 + i\sqrt{2}, \ -1 - i\sqrt{2}$

10. Define the locus of the following points;

i) $|z + 2i| = |2zi - 1|$ ii) $\left|\dfrac{z+i}{z-2}\right| = 3$

ANALYSIS

1. Sketch the curve $y = \dfrac{x^3}{9 - x^2}$, clearly stating the asymptotes.

Intercepts: when $y = 0, \quad x = 0$ so, $(0, \ 0)$, thus curve crosses the axes at origin.

Vertical asymptotes: $9 - x^2 = 0$, so vertical asymptotes at $x = -3, \quad x = 3$

To find the slant asymptotes: we perform long division:

$$\begin{array}{r|l} & -x \\ \hline -x^2-9 & x^3+0x^2+0x+0 \\ & -\left(x^3-9x\right) \\ \hline & 9x \end{array}$$

We get $y = -x + \dfrac{9x}{9-x^2}$, thus as $x \to \pm\infty$, $y \to -x$, so the line $y = -x$ is the slant asymptote.

For the turning points, we have $\dfrac{dy}{dx} = 0$,

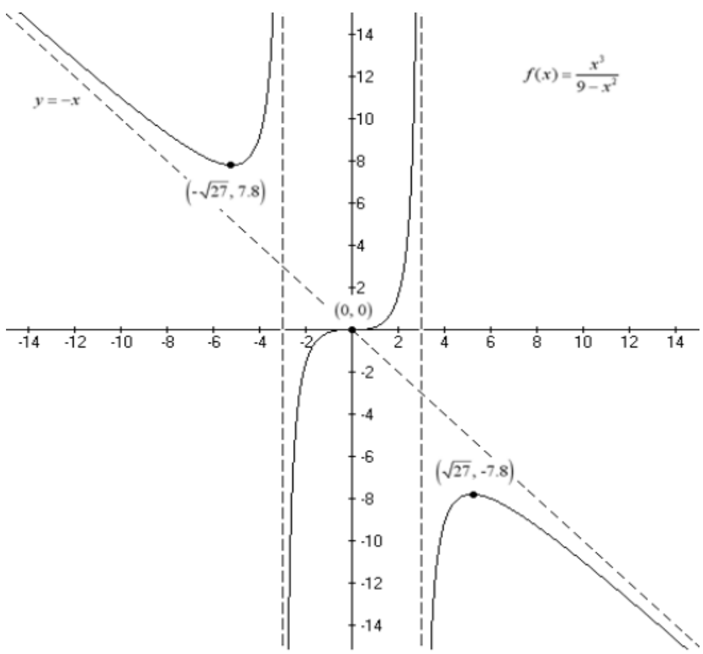
$$\frac{dy}{dx} = \frac{3x^2(9-x^2) - x^3 . -2x}{(9-x^2)^2} = 0$$

$$27x^2 - 3x^4 + 2x^4 = 0$$

$x^2(27 - x^2) = 0$ these occur when $x = 0$, $x = \pm\sqrt{27}$,

$x = 0,\ y = 0$, $x = 3\sqrt{3},\ y = -7.794$, $x = -3\sqrt{3},\ y = 7.794$

i.e curve has a local minimum at $(-5.196\ ,\ 7.794)$ and a local maximum at $(5.196\ ,\ -7.794)$ and a negative point of inflexion at $(0,\ 0)$



2. Show that $e^{x}\cos x$ has turning points at intervals of π in x. Distinguish between the maximum and minimum and show that these values are in a geometrical progression with common ratio $-e^{\pi}$.

$\frac{dy}{dx}=-e^{-x}\cos x+e^{-x}\sin x$, for t.p $\frac{dy}{dx}=0$,

so $e^{-x}(\cos x+\sin x)=0$, thus $\tan x=-1$, $x=\frac{3}{4}\pi,\ \frac{7}{4}\pi,\ \frac{11}{4}\pi$

$$x=\frac{3}{4}\pi,\quad y=e^{-\frac{3}{4}\pi}\cos\frac{3\pi}{4}\ =\ y=-\frac{\sqrt{2}}{2}e^{-\frac{3}{4}\pi}$$

$$x=\frac{7}{4}\pi,\quad y=e^{-\frac{7}{4}\pi}\cos\frac{7\pi}{4}\ =\ y=\frac{\sqrt{2}}{2}e^{-\frac{7}{4}\pi}$$

$$x=\frac{11}{4}\pi,\quad y=e^{-\frac{11}{4}\pi}\cos\frac{11\pi}{4}=\ y=-\frac{\sqrt{2}}{2}e^{-\frac{11}{4}\pi}$$

$r=\dfrac{e^{-\frac{7\pi}{4}}}{-e^{-\frac{3\pi}{4}}}=-e^{-\pi}$ or $r=\dfrac{-e^{-\frac{11\pi}{4}}}{e^{-\frac{7\pi}{4}}}=-e^{-\pi}$ thus values of y form a geometric progression with common ratio $-e^{-\pi}$.

3. A curve is given parametrically by $x=2\cos\theta+\cos 2\theta,\ y=2\sin\theta-\sin 2\theta.$ Show that the gradient at the point parameter θ is $-\tan\frac{1}{2}\theta$ and that the equation of the tangent to the curve at this point is $x\sin\frac{1}{2}\theta+y\cos\frac{1}{2}\theta=\sin\frac{3}{2}\theta.$

$\frac{dx}{d\theta}=-2(\sin\theta+\sin 2\theta)$, $\qquad \frac{dy}{d\theta}=2(\cos\theta-\cos 2\theta)$, thus,

$$\frac{dy}{dx}=\frac{2(\cos\theta-\cos 2\theta)}{-2(\sin\theta+\sin 2\theta)}\ ,\qquad \frac{dy}{dx}=\frac{-2\sin\frac{3}{2}\theta\sin-\frac{1}{2}\theta}{2\sin\frac{3}{2}\theta\cos-\frac{1}{2}\theta}=-\tan\frac{1}{2}\theta$$

Equation of the tangent is

$$\frac{y-2\sin\theta+\sin 2\theta}{x-2\cos\theta-\cos 2\theta}=-\frac{\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}$$

$$y\cos\tfrac{1}{2}\theta-2\sin\theta\cos\tfrac{1}{2}\theta+\sin 2\theta\cos\tfrac{1}{2}\theta=-x\sin\tfrac{1}{2}\theta+2\cos\theta\sin\tfrac{1}{2}\theta+\cos 2\theta\sin\tfrac{1}{2}\theta$$

$$y\cos\tfrac{1}{2}\theta+x\sin\tfrac{1}{2}\theta=\sin\tfrac{3}{2}\theta-\sin\tfrac{1}{2}\theta+\tfrac{1}{2}(\sin\tfrac{5}{2}\theta-\sin\tfrac{3}{2}\theta)+\sin\tfrac{3}{2}\theta+\sin\tfrac{1}{2}\theta-\tfrac{1}{2}(\sin\tfrac{5}{2}\theta+\sin\tfrac{3}{2}\theta)$$

$y\cos\frac{1}{2}\theta + x\sin\frac{1}{2}\theta = \sin\frac{3}{2}\theta$ as required.

4. Show that the particular solution to the equation $x - y\frac{dy}{dx} = y^2\frac{dy}{dx} + xy$, for $y(0) = 2$, is $x^2 + (y-2)(y+6) + 4\ln(y-1) = 0$.

$$x - xy = (y^2 + y)\frac{dy}{dx}, \text{ thus } \quad -\int x\,dx = \int \frac{y^2 + y}{y-1}\,dy$$

$$-\int x\,dx = \int (y+2)\,dy + 2\int \frac{1}{y-1}\,dy$$

$$-\frac{x^2}{2} = \frac{y^2}{2} + 2y + 2\ln(y-1) + C, \text{ for } x = 0, \quad y = 2$$

$$0 = 2 + 4 + \ln 1 + C \;, \therefore\; C = -12$$

$$-x^2 = y^2 + 4y - 12 + 4\ln(y-1)\;,$$

$$x^2 + (y-2)(y+6) + 4\ln(y-1) = 0$$

5. A sample of a radioactive radium losses mass at a rate which is proportional to the amount present. If M is the mass after time t years,

(i) Form a differential equation connecting M and t.

(ii) If initially the mass of radium is M_o, deduce that $M = M_o e^{-kt}$.

(iii) Given that its initial mass halved in 6000 years, find the value of the constant k, hence determine the number of years it takes for $5g$ of radium to reduce to $3.6g$.

$$\frac{dM}{dt} \propto M \;\Rightarrow\; \frac{dM}{dt} = -kM$$

$$\Rightarrow \int \frac{dM}{M} = -k\int dt, \text{ so } \ln M = -kt + C \text{ when } t = 0, \; M = M_o \text{ thus } \ln M_o = C$$

$$\ln M - \ln M_o = -kt \quad \text{or} \quad \ln\left(\frac{M}{M_o}\right) = -kt \text{ thus } \frac{M}{M_o} = e^{-kt} \;\therefore\; M = M_o e^{-kt}$$

When $t = 1600\ yrs$, $M = \frac{1}{2}M_o$

$$\Rightarrow \quad \frac{1}{2}M_o = M_o e^{-1600k} \text{ so } 1600k = \ln 2 \quad k = \frac{\ln 2}{1600} \approx 0.00043322$$

When $M_o = 5$, $M = 3.6g$

$$t = \frac{1}{k} In \frac{M_o}{M} \qquad t = \frac{1600}{In\, 2} In \frac{5}{3.6} \approx 758.29 \text{ years}$$

6. Prove that $\int_1^3 \left(\frac{3-x}{x-1}\right)^{1/2} dx = \pi$ (hint: Use the substitution $x = 3\sin^2\theta + \cos^2\theta$.

$$\int_1^3 \left(\frac{3-3+2\cos^2\theta}{3-2\cos^2\theta - 1}\right)^{1/2} .4\sin\theta\cos\theta . d\theta$$

$x = 3 - 2\cos^2\theta$

$dx = 4\sin\theta\cos\theta\, d\theta$

$$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{2}\cos\theta}{\sqrt{2}\sin\theta} .4\sin\theta\cos\theta\, d\theta$$

$x = 1, \quad \theta = 0$

$x = 3, \quad \theta = \frac{\pi}{2}$

$$= 4\int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta \qquad = 2\int_0^{\frac{\pi}{2}} (1 + \cos 2\theta)\, d\theta \qquad = \quad 2\left[\left(\theta + \frac{1}{2}\sin 2\theta\right)\right]_0^{\frac{\pi}{2}}$$

$$= 2\left[\left(\frac{\pi}{2} - 0\right) - (0)\right] \qquad = \pi$$

7. Evaluate: i) $\int_0^1 \frac{6x}{x^3+8} dx$ ii) $\int_0^3 x \log_e (1+x) dx$

8. Evaluate:

i) $\int_0^{\frac{a}{2}} x^2 \sqrt{(a^2 - x^2)} dx$

$$\int_0^{\frac{a}{2}} x^2 \sqrt{(a^2 - x^2)} dx = \int_0^{\frac{a}{2}} x^2 \sqrt{a^2\left(1 - \frac{x^2}{a^2}\right)} dx$$

Let $\frac{x}{a} = \sin\theta : dx = a\sin\theta\, d\theta$

$x = a\sin\theta$

$$= \int_0^{\frac{a}{2}} a.(a\sin\theta)^2 \sqrt{(1-\sin^2\theta)}.\, a\sin\theta\, d\theta$$

$x = 0, \quad \theta = 0$

$x = \frac{a}{2}, \quad \theta = \frac{\pi}{6}$

$$= a^4 \int_0^{\frac{\pi}{6}} \sin^2\theta\cos^2\theta\, d\theta$$

$\sin 2\theta = 2\sin\theta\cos\theta$

$$= \frac{a^4}{4}\int_0^{\frac{\pi}{6}} \sin^2 2\theta\, d\theta$$

$\sin^2 2\theta = \frac{1}{2}(1 - \cos 4\theta)$

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$$= \frac{a^4}{8}\int_0^{\frac{\pi}{6}}(1-\cos 4\theta)\,d\theta \qquad = \frac{a^4}{8}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{6}}$$

$$= \frac{a^4}{8}\left[\left(\frac{\pi}{6} - \frac{1}{4}\sin\frac{2\pi}{3}\right) - 0\right] \qquad = \frac{a^4}{8}\left(\frac{\pi}{6} - \frac{\sqrt{3}}{8}\right)$$

ii) $\displaystyle\int_0^{\pi/2}\frac{dx}{2+\cos x}$ $\qquad \cos x = \dfrac{1-t^2}{1+t^2}$ $\qquad$ Let $t = \tan\dfrac{x}{2}$, $\dfrac{1}{2}(1+t^2)dx = dt$

$x = 0,\ t = 0, \qquad x = \frac{\pi}{2},\ t = 1$

$$\int_0^{\pi/2}\frac{dx}{2+\cos x} = \int_0^1 \frac{\frac{2\,dt}{(1+t^2)}}{\frac{(3+t^2)}{(1+t^2)}} = 2\int_0^1\frac{dt}{3+t^2}$$

$$2\left[\frac{1}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}}\right]_0^1 = 2\frac{1}{\sqrt{3}}\left(\tan^{-1}\frac{1}{\sqrt{3}} - \tan^{-1}0\right) = \frac{2}{\sqrt{3}}.\frac{\pi}{6} = \frac{\pi}{3\sqrt{3}}$$

ALTERNATIVELY: $\displaystyle\int_0^1\frac{2\,dt}{3\left(1+\left(\frac{t}{\sqrt{3}}\right)^2\right)}$ $\qquad \dfrac{t}{\sqrt{3}} = \tan\theta,\quad \dfrac{1}{\sqrt{3}}dt = \sec^2\theta\,d\theta$

$t = 0,\ \theta = 0 \qquad t = 1,\ \theta = \pi/6$

$$\frac{2}{3}\int_0^{\pi/6}\sqrt{3}\,d\theta \quad = \frac{2\sqrt{3}}{3}[\theta]_0^{\pi/6} = \frac{2\sqrt{3}}{3}\left(\frac{\pi}{6} - 0\right) = \frac{\sqrt{3}}{9}\pi$$

iii) $\displaystyle\int_0^1\frac{x^2}{\sqrt{4-x^2}}\,dx$

9. Use:

i) $t = \tan x$ find $\displaystyle\int\frac{dx}{\cos^2 x + 4\sin^2 x}$

$$\int\frac{dx}{\cos^2 x + 4\sin^2 x} = \int\frac{\sec^2 x\,dx}{1+4\tan^2 x} \qquad t = \tan x,\quad dt = \sec^2 x\,dx$$

$$= \int\frac{\sec^2 x}{1+4t^2}.\frac{dt}{\sec^2 x} \qquad = \int\frac{dt}{1+4t^2}$$

$= \frac{1}{2}\tan^{-1} 2t + c \qquad = \frac{1}{2}\tan^{-1} 2(\tan x) + c$

ii) $\int_0^1 \frac{e^x}{(1+e^x)}\,dx$, use $t = e^x$

10. Evaluate: i) $\int_0^{\pi/2} \frac{1}{5\cos x + 4}\,dx$, use $t = \tan\frac{x}{2}$

$\int_0^{\frac{\pi}{2}} \frac{dx}{5\cos x + 4}$

$\cos x = \frac{1-t^2}{1+t^2}$, $t = \tan\frac{1}{2}x$, $dt = \frac{1}{2}\sec^2\frac{1}{2}x\,dx$

$\int_0^1 \frac{(1+t^2)}{4 + 5\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2\,dt}{1+t^2}$

$x = \frac{\pi}{2}, \quad t = 1$

$x = 0, \quad t = 0$

$\int_0^1 \frac{1}{9-5t^2}\,dt$

let $\frac{1}{9-5t^2} \equiv \frac{A}{2+t} + \frac{B}{2-t}$

$\frac{1}{4}\int_0^1 \left(\frac{1}{2+t} + \frac{1}{2-t}\right) dt$

$\frac{1}{4}\left[ln\,\frac{2+t}{2-t}\right]_0^1 = \frac{1}{4}[ln\,3 - ln\,1] = \frac{1}{4}\,ln\,3$

ii) $\int_1^2 \frac{dx}{\sqrt{12 + 8x - 4x^2}}$

iii) $\int_{\pi/3}^{\pi/2} \frac{1}{1 + \cos x + \sin x}\,dx$ use $t = \tan\frac{x}{2}$

11. Prove that $\int \sec^n x\,dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$

12. If $x^2 + 2xy + 3y^2 = 1$ prove that $(x+3y)^3 \frac{d^2y}{dx^2} + 2 = 0$

13. Find the solution to the differential equation: $\frac{dv}{dt} = 9.8 - 0.196\,v$

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$\frac{dv}{dt} = 9.8 - 0.196\,v$, $\frac{dv}{dt} + 0.196\,v = 9.8$

$I.F = e^{\int 0.196\,dt} = e^{0.196\,t}$

Thus, $\frac{d(e^{0.196\,t}v)}{dt} = 9.8e^{0.196\,t} \Rightarrow \int d(e^{0.196\,t}v) = \int 9.8e^{0.196\,t}\,dt$

$e^{0.196\,t}v = 50\,e^{0.196\,t} + C$ $\qquad$ $v = 50 + Ce^{-0.196\,t}$

TRIGONOMETRY

1. Prove that: i) $\tan^{-1}\frac{\sqrt{3}}{2} + \tan^{-1}\frac{\sqrt{3}}{5} = \frac{\pi}{3}$ $\qquad$ ii) $\tan^{-1}\frac{1}{2} - \cos ec^{-1}\frac{\sqrt{5}}{2} = \cos^{-1}\frac{4}{5}$

2. Prove that. $\frac{\tan A - 3\tan 3A}{\cot A - 3\cot 3A} = \frac{\tan A - 3\cot A}{\cot A - 3\tan A}$

From L.H.S $\frac{\tan A - 3\tan 3A}{\cot A - 3\cot 3A} = \dfrac{\tan A - \dfrac{3(3\tan A - \tan^3 A)}{1 - 3\tan^2 A}}{\cot A - \dfrac{3(\cot^3 A - 3\cot A)}{3\cot^3 A - 1}}$

$$= \dfrac{\dfrac{\tan A - 3\tan^3 A - 9\tan A + 3\tan^3 A}{1 - 3\tan^2 A}}{\dfrac{3\cot^3 A - 3\cot^3 A - \cot A + 9\cot A}{3\cot^2 A - 1}}$$

$$= \frac{-8\tan A}{1 - 3\tan^2 A}\cdot\frac{3\cot^2 A - 1}{8\cot A} \qquad = \frac{\tan A - 3\cot A}{\cot A - 3\tan A}$$ as the R.H.S

3. Prove that in any triangle PQR, $\frac{1}{p}\cos^2\tfrac{1}{2}P + \frac{1}{q}\cos^2\tfrac{1}{2}Q + \frac{1}{r}\cos^2\tfrac{1}{2}R = \frac{(p+q+r)^2}{4pqr}$

4. Prove that $\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$, and $\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$ and hence solve the equation $t^4 + 4t^3 - 6t^2 - 4t + 1 = 0$.

5. Given that in a triangle MNB, $\tan A = \frac{n+b}{n-b}\tan\tfrac{1}{2}M$, where n, b and m are sides of the triangle MNB, show that $m = (n-b)\sec A\cos\tfrac{1}{2}M$.

6. Prove that: $8\cos 3\theta \cos 2\theta \cos \theta - 1 = \dfrac{\sin 7\theta}{\sin \theta}$.

7. Prove that $\dfrac{a+b-c}{a+b+c} = \tan\frac{1}{2}A\tan\frac{1}{2}B$

R.H.S $\tan\frac{1}{2}A\tan\frac{1}{2}B = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}} \times \sqrt{\dfrac{(s-a)(s-c)}{s(s-b)}}$

$$= \sqrt{\frac{(s-a)(s-b)(s-c)^2}{s^2(s-a)(s-b)}} = \frac{s-c}{s}$$

$$= \frac{\frac{a+b+c}{2} - c}{\frac{a+b+c}{2}} = \frac{a+b-c}{a+b+c} \text{ as L.H.S}$$

ALT: From L.H.S.

$$\frac{a+b-c}{a+b+c} = \frac{2R(\sin A + \sin B - \sin C)}{2R(\sin A + \sin B + \sin C)}$$

$$= \frac{2\sin\frac{1}{2}A\cos\frac{1}{2}A + 2\cos\frac{(B+C)}{2}\sin\frac{(B-C)}{2}}{2\sin\frac{1}{2}A\cos\frac{1}{2}A + 2\sin\frac{(B+C)}{2}\cos\frac{(B-C)}{2}}$$

$\cos\frac{B+C}{2} = \cos(90 - \frac{A}{2}) = \sin\frac{1}{2}A$

$\sin\frac{B+C}{2} = \cos(90 - \frac{A}{2} = \cos\frac{1}{2}A)$

$$= \frac{\sin\frac{1}{2}A\cos\frac{1}{2}A + \sin\frac{A}{2}\sin\frac{(B-C)}{2}}{\sin\frac{1}{2}A\cos\frac{1}{2}A + \cos\frac{A}{2}\cos\frac{(B-C)}{2}} = \frac{\sin\frac{1}{2}A(\cos\frac{1}{2}A + \sin\frac{B-C}{2})}{\cos\frac{1}{2}A(\sin\frac{1}{2}A + \cos\frac{B-C}{2})}$$

$$= \frac{\sin\frac{1}{2}A(\sin\frac{B+C}{2} + \sin\frac{B-C}{2})}{\cos\frac{1}{2}A(\cos\frac{B+C}{2} + \cos\frac{B-C}{2})} = \frac{\sin\frac{1}{2}A(2\sin\frac{B}{2}\cos\frac{C}{2})}{\cos\frac{1}{2}A(2\cos\frac{B}{2}\cos\frac{C}{2})}$$

$= \tan\frac{1}{2}A\tan\frac{1}{2}B$ as required.

8. Prove that $\tan^2\left(\dfrac{\pi}{4} + \theta\right) = \dfrac{1+\sin 2\theta}{1-\sin 2\theta}$, prove also that $\tan 22\frac{1}{2}^o = \sqrt{2} - 1$ and $\tan 67\frac{1}{2}^o = \sqrt{2} + 1$.

9. Prove that if $\tan x = k\tan(A - x)$, then $\sin(2x - A) = \dfrac{k-1}{k+1}\sin A$.. Find all the angles for $0^o \le x \le 360^o$ which satisfy the equation $2\tan x - \tan(30^o - x) = 0$.

VECTORS

1. The point $C(a, 4, 5)$ divides the line joining $A(1, 2, 3)$ and $B(6, 7, 8)$ in the ratio $\lambda : 3$.

Find a and λ.

2. Show that the lines $r=(-2i+5j-11k)+\lambda(3i+j+3k)$, $r=(8i+9j)+t(4i+2j+5k)$ intersect, hence, find the position vector of their point of intersection. Find also the Cartesian equation of the plane formed by these two lines.

3a) Determine the equation of the plane through the points $A(1,\ 1,\ 2)$, $B(2,\ -1,\ 3)$ and $C(-1,\ 2,\ -2)$.

b) A line through the point $D(-13,\ 1,\ 2)$ and parallel to the vector $12\mathbf{i}+6\mathbf{j}+3\mathbf{k}$ meets the plane in (a) at point E. Find:
i) the coordinates of E.
ii) The angle between the line and the plane.

4. The points A, B, C and D have coordinates $(-7,\ 9)$, $(3,\ 4)$, $(1,\ 2)$ and $(-2,\ -9)$ respectively. Find the vector equation of the line PQ where P divides AB in the ratio $2:3$ and Q divides CD in the ratio $1:-4$.

$$\vec{AP}=\frac{2}{5}\left[\begin{pmatrix}3\\4\end{pmatrix}-\begin{pmatrix}-7\\9\end{pmatrix}\right]=\begin{pmatrix}4\\-2\end{pmatrix}$$

$$\vec{OP}=\vec{OA}+\vec{AP},\quad \vec{OP}=\begin{pmatrix}-7\\9\end{pmatrix}+\begin{pmatrix}4\\-2\end{pmatrix}=\begin{pmatrix}-3\\7\end{pmatrix}$$

$$\vec{CQ}=\frac{1}{-3}\left[\begin{pmatrix}-2\\-9\end{pmatrix}-\begin{pmatrix}1\\2\end{pmatrix}\right]=\begin{pmatrix}1\\ 11/3\end{pmatrix}$$

$$\vec{OQ}=\vec{OC}+\vec{CQ},\quad \vec{OQ}=\begin{pmatrix}1\\2\end{pmatrix}+\begin{pmatrix}1\\ 11/3\end{pmatrix}=\begin{pmatrix}2\\ 17/13\end{pmatrix}$$

$r=a+\lambda d$.

$$d=\begin{pmatrix}2\\ 17/3\end{pmatrix}-\begin{pmatrix}-3\\-1\end{pmatrix}=\begin{pmatrix}5\\ -4/3\end{pmatrix},\ \text{thus}\ r=\begin{pmatrix}-3\\7\end{pmatrix}+\lambda\begin{pmatrix}4\\ -4/3\end{pmatrix}$$

5. Given that A, B and C are three collinear points whose position vectors are a, b, c respectively, satisfy the equation $\alpha a+\beta b+\gamma c=0$, where α, β, γ are scalars, prove that $\alpha+\beta+\gamma=0$.

6. A plane passes through the point $(1,2,3)$ and is perpendicular to the vector $i-5j+4k$. The plane meets the $z-$plane in P and the $y-$plane in Q. Find:

i) the equation of the plane. ii) the distance PQ.

7.a) Show that the lines $\frac{x-1}{2}=\frac{y+1}{3}=z$ and $\frac{x+1}{5}=\frac{2-y}{-1}=\frac{2-z}{-1}$ do not intersect. Hence, find the shortest distance between them.

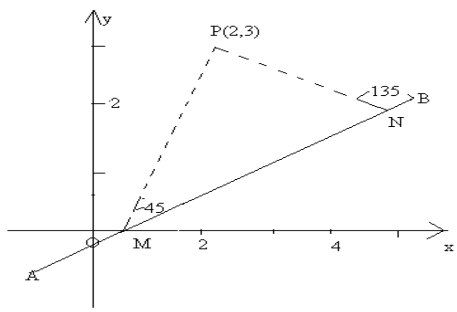
b) Find the equation of the line of intersection of the planes, $2x+3y+4z=1$ and $x+y+3z=0$.

8a) Show that the vectors $a=2i+3j+4k$, $b=-i+2j-3k$, $c=i+5j+k$ form a triangle and determine the area of this triangle.

GEOMETRY

1. Find the equations to the lines through the point (2, 3) which makes angles of 45° with the line $x-2y=1$.

The equation $x-2y=1$ has gradient $\frac{1}{2}$.



Its clear from the diagram that there are two possible lines PM, PN which makes an angle 45° with AB, and the tangents of these angles are respectively,

$\tan 45^o=1$ and $\tan 135^o=-1$, hence, if m is the slope of PM, gives

$$1=\frac{m-(1/2)}{1+(m/2)},\quad m=3$$

Similarly, if m' is the slope of PN, then

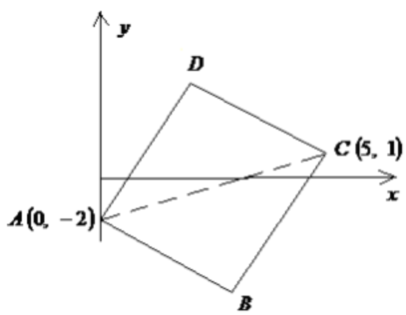
$$-1=\frac{m'-(1/2)}{1+(m'/2)},\quad m'=-1/3$$

The required lines that pass through (2, 3) have slopes 3 and $-1/3$, thus the equations are;

$$y-3=3(x-2) \quad \text{or} \quad 3x-y=3$$

$y-3=-\frac{1}{3}(x-2)$ or $x+3y=11$.

2. $ABCD$ is a square; A is the point $(0, \ -2)$ and C is the point $(5, \ 1)$, AC being the diagonal. Find the equations of the lines AB and BC.



AD & AB make angles of 45^o with diagonal AC.

So, the gradient of $AC = \dfrac{1-(-2)}{5-0}=\dfrac{3}{5}$, so if m_1 & m_2 are the gradients of AD & AB,

then, $\tan 45 = \dfrac{m_1 - \frac{3}{5}}{1+\frac{3}{5}m_1}$, where $m_1 = 4$ and so $m_2 = -\dfrac{1}{4}$ since AD & AB are perpendicular.

Equation of AD is; $\dfrac{y+2}{x-0}=4$, $y=4x-2$

Equation of AB is; $\dfrac{y+2}{x-0}=-\dfrac{1}{4}$, $4y+x+8=0$

3. The line $y=mx$ and the curve $y=x^2-2x$ intersect at the origin O and meet again at a point A. If P is the midpoint of OA, find the locus of P.

4. A circle with centre P and radius r touches externally both the circles $x^2+y^2=4$ and $x^2+y^2-6x+8=0$. Prove that the $x-$coordinate of P is $\dfrac{r}{3}+2$.

$x^2+y^2=4$ has centre = $(0,0)$ and $r_1=2$

For $x^2+y^2-6x+8=0$ has centre = $(3,,0)$ and $r_2=1$

If they touch externally, $C_1C_2=r+2$, $C_2C_3=r+1$

If $C_2(a,b)$ then $\Rightarrow$ $\sqrt{(a^2+b^2)}=2+r$

And $\sqrt{(3-a)^2+b^2} = r+1$

We get $a^2+b^2=(2+r)^2=4+4r+r^2$ .... (3)

And ...... (4)

Eqn(3)-eqn(4), we get

$-9+6a=3+2r$ thus $6a=2r+12$

$\therefore$ $x-$coordinate, $a=\frac{r}{3}+2$

5. Determine the equations of the tangents to the parabola $y^2=6x$ from the point $(2,\ 4)$.

Using $y=mx+\frac{a}{m}$, comparing $y^2=6x$ and $y^2=4ax$, $a=\frac{3}{2}$.

So, $y=mx+\frac{3}{2m}$, passés through $(2,\ 4)$, thus, $4=2m+\frac{3}{2m}$

$4m^2-8m+3=0,\quad (2m-3)(2m-1)=0,\quad m=-\frac{3}{2},\ \frac{1}{2}$

$\therefore$ the equations of the tangents are: $y=\frac{3}{2}x+1,\quad y=\frac{1}{2}x+3$

6. If the tangents at points P and Q on the parabola $y^2=4ax$ are perpendicular, find the locus of the mid point of PQ.

Let P and Q be the points $(at_1^{\ 2}, 2at_1)$, $(at_2^{\ 2}, 2at_2)$.

Gradient of tangent at P = $\frac{1}{t_1}$, Gradient of tangent at Q = $\frac{1}{t_2}$

Hence since tangents are perpendicular, then, $\frac{1}{t_1}.\frac{1}{t_2}=-1$ i.e $t_1t_2=-1$

Let the coordinates of the mid point of PQ be (α, β)

Thus; $\alpha=a\left(\frac{t_1^{\ 2}+t_2^{\ 2}}{2}\right)$ and $\beta=a(t_1+t_2)$

To eliminate t_1 and t_2 square the second equation, giving

$$\beta^2=a^2(t_1+t_2)^2=a^2(t_1^{\ 2}+t_2^{\ 2}+2t_1t_2)$$

$$= 2a.\frac{a}{2}(t_1^{\ 2} + t_2^{\ 2}) + 2a^2(-1) \text{ since } t_1 t_2 = -1$$

$$= 2a\alpha - 2a^2$$

So the locus of the midpoint of PQ has the equation, $y^2 = 2a(x - a)$

MATH SEMINAR algebra

10(ii) $\left|\frac{z+i}{z-2}\right| = 3$

$$let\ z = x + yi$$

$$\left|\frac{x + yi + i}{x + yi - 2}\right| = 3$$

$$\left|\frac{x+(y+1)i}{x-2+yi}\right| = 3$$

$$\frac{\sqrt{x^2+(y+1)^2}}{\sqrt{(x-2)^2+y^2}} = 3$$

$$\frac{x^2+y^2+2y+1}{x^2-4x+4+y^2} = 9$$

$$x^2 + y^2 + 2y + 1 = 9x^2 + 9y^2 - 36x + 36$$

$$8x^2 + 8y^2 - 36x - 2y + 35 = 0$$

$$x^2 + y^2 - \frac{9}{2}x - \frac{1}{4}y + \frac{35}{8} = 0$$

$$\left(x - \frac{9}{4}\right)^2 - \frac{81}{16} + \left(y - \frac{1}{8}\right)^2 - \frac{1}{64} + \frac{35}{8} = 0$$

$$\left(x - \frac{9}{4}\right)^2 + \left(y - \frac{1}{8}\right)^2 \frac{45}{64} = 0$$

$$equation\ of\ a\ circle\ with\ centre \left(\frac{9}{4}, \frac{1}{8}\right) and\ radii \sqrt{\frac{45}{64}}$$

MATH SEMINAR algebra

"FAILURE IS NOT AN OPTION BUT IT IS A PREVAILING ATTITUDE"

10(i) $|z+2i| = |2zi-1|$

Let $z = x + yi$

$|x+yi+2i| = |2(x+yi)-1|$

$|x+(y+2)i| = |(2x-1)+2yi|$

$\sqrt{x^2+(y+2)^2} = \sqrt{(2x-1)^2+(2y)^2}$

$x^2+y^2+4y+4 = 4x^2-4x+1+4y^2$

$(4x^2-x^2)+(4y^2-y^2)-4x-4y+1-4=0$

$3x^2+3y^2-4x-4y-3=0$

$x^2-\frac{4}{3}x+y^2-\frac{4}{3}y-3$

$\left(x-\frac{2}{3}\right)^2-\frac{4}{9}+\left(y-\frac{2}{3}\right)^2-\frac{4}{9}-3=0$

$\left(x-\frac{2}{3}\right)^2+\left(y-\frac{2}{3}\right)^2-\frac{35}{9}=0$

Equation of a circle with centre$\left(\frac{2}{3},\frac{2}{3}\right)$ $and\ radii\ \sqrt{\frac{35}{9}}$

MATH SEMINAR **analysis**

8(iii) $\int_0^1 \frac{x^2}{\sqrt{4-x^2}}$

$\int \frac{x^2}{\sqrt{4-x}} = \int \frac{x^2}{\sqrt{4\left[1-\left(\frac{x}{2}\right)^2\right]}}$

Let $\frac{x}{2}$ be =sin u

Type equation here.

$\cos u\,\frac{\delta u}{\delta x} = \frac{1}{2}$

$\delta x = 2\cos u\,\delta u$

$$\int \frac{x^2}{2\sqrt{1-sin^2u}} \cdot 2\cos u \cdot du$$

But $\sqrt{1-sin^2u} = \cos u$

$$\int \frac{x^2}{\cancel{2cosu}} \cdot \cancel{2cosu} \cdot du$$

$$\int x^2 \cdot du$$

But x= $2\sin u$

$$x^2 = 4sin^2u$$

$$\cos 2u = 1 - 2sin^2u$$

$$2sin^2u = 1 - cos2u$$

$$\int 4sin^2u \cdot du$$

$$2\int 2sin^2u$$

$$2[\int (1-\cos 2u)] \cdot du$$

$$2[\int du - \int \cos 2u]$$

$$2\left[u - \frac{\sin 2u}{2}\right] + c$$

$$\int_0^1 \frac{x^2}{\sqrt{4-x^2}} = 2\left[\sin^{-1}\frac{x}{2} - \sin 2\left(\sin^{-1}\frac{x}{2}\right)\right]_0^1$$

$$2\left[\left(\sin^{-1}\frac{1}{2} - \sin 2\left(\sin^{-1}\frac{1}{2}\right)\right) - \left(\sin^{-1} 0 - \sin 2(\sin^{-1} 0)\right)\right]$$

$$2[30 - \sin 60]$$

58.26794919

<u>58.268</u>